# Giovedì 27 ottobre 8:30-11:00 (canale B)

Titolo nota 27/10/2022

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & -2 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$$p_D(\lambda) = \lambda^2 \cdot (\lambda-2)^2 \, ; \quad p_E(\lambda) = \lambda^2 \cdot (\lambda-2)^2 \, ; \quad p_F(\lambda) = (\lambda-3)^4$$

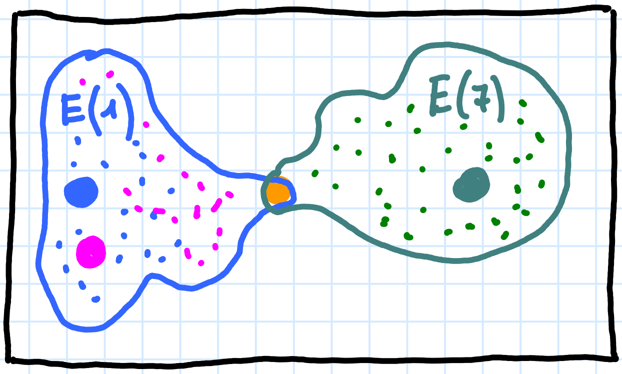
---

$$\underset{3\times 3}{C} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$\lambda_1 = 1 \qquad x\cdot(1,0,-1) + y\cdot(0,1,-1)$

$\lambda_2 = 7 \qquad x\cdot(1,2,3)$

$\mathbb{R}^3$




- $(0,0,0)$
- $(1,0,-1)$
- $(0,1,-1)$ indip
- $(1,2,3)$

$\mathcal{B} = \big((1,0,-1), (0,1,-1), (1,2,3)\big)$ è una base di $\mathbb{R}^3$ formata da autovettori di $C$

verifica: $\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} = 6 \neq 0 \Rightarrow$ rg MAX $\Rightarrow$ lin. indip.

$\exists$ base di $\mathbb{R}^3$ formata da autovettori di $C$

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 6 & 0 & 4 \end{bmatrix} \quad \begin{array}{ll} \lambda_1 = 0 & (-2,0,3) \\ \lambda_2 = 1 & (0,1,0) \\ \lambda_3 = 7 & (1,0,2) \end{array}$$

$\exists\, \mathcal{B} = ((-2,0,3),(0,1,0),(1,0,2))$ base di $\mathbb{R}^3$ formata da autovettori di A

$$B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} \quad \begin{array}{ll} \lambda_1 = 2 & (-1,1,1) \\ \lambda_2 = 4 & (1,-1,1) \end{array}$$

$\nexists\, \mathcal{B}$ base di $\mathbb{R}^3$ formata da autovettori di B

$$C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \quad \begin{array}{ll} \lambda_1 = 1 & (1,0,-1),(0,1,-1) \\ \lambda_2 = 7 & (1,2,3) \end{array}$$

$\exists\, \mathcal{B} = ((1,0,-1),(0,1,-1),(1,2,3))$ formata da autovettori di C

---

## LEMMA

$A_{n \times n}$ ; $\lambda_1, \lambda_2, \dots, \lambda_m$ $\quad m \leq n$

autovalori reali a due a due distinti

Esiste una base di $\mathbb{R}^n$ formata da autovettori di A

se e solo se

$$\sum_{i=1}^{m} \underbrace{\dim(E(\lambda_i))}_{\parallel \text{ DEF.}} = n$$

$$\sum_{i=1}^{m} m_g(\lambda_i) = n$$

---

## TEOREMA (senza dimostrazione)

$A_{n \times n}$

$\lambda_1, \lambda_2, \lambda_3, \dots\dots, \lambda_m$ $\quad m \leq n$

autovalori reali a due a due distinti.

Esiste una base di $\mathbb{R}^n$ formata da autovettori di A

se e solo se

valgono le seguenti due condizioni:

1. le radici del polinomio caratteristico di A sono TUTTE reali;

② per ogni autovalore $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ si ha che

$$m_g(\lambda_i) = m_a(\lambda_i) \quad i=1,2,3,\dots,m$$

---

$A_{5\times5}$ $\quad p_A(\lambda) = (\lambda-2)^2(3-\lambda)^1 \cdot \underbrace{(\lambda^2+1)}_{}$ → NON ha radici reali

$$m_a(2) = 2 \Rightarrow m_g(2) \leq 2$$
$$m_a(3) = 1 \Rightarrow m_g(3) = 1$$

$m_g(2) + m_g(3) \leq 3$ non sarà mai 5

$\nexists$ base di $\mathbb{R}^5$ formata da autovettori di $A$

---

$$m_g(\lambda_i) \leq m_a(\lambda_i)$$

TUTTE RADICI REALI

$$\sum_{i=1}^{m} m_g(\lambda_i) \leq \sum_{i=1}^{m} m_a(\lambda_i) = n$$

$$\sum_{i=1}^{m} m_g(\lambda_i) = n \iff m_g(\lambda_i) = m_a(\lambda_i) \quad i=1,2,3,\dots,m$$

---

$$A=\begin{bmatrix}3&0&2\\0&1&0\\6&0&4\end{bmatrix}\qquad \begin{matrix}\lambda_1=0 & (-2,0,3)\\ \lambda_2=1 & (0,1,0)\\ \lambda_3=7 & (1,0,2)\end{matrix}$$

$$B=\begin{bmatrix}2&-1&1\\0&3&-1\\2&1&3\end{bmatrix}\qquad \begin{matrix}\lambda_1=2 & (-1,1,1)\\ \lambda_2=4 & (1,-1,1)\end{matrix}$$

$$C=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}\qquad \begin{matrix}\lambda_1=1 & (1,0,-1),\ (0,1,-1)\\ \lambda_2=7 & (1,2,3)\end{matrix}$$

---

$$A=\begin{bmatrix}3&0&2\\0&1&0\\6&0&4\end{bmatrix}\qquad \begin{matrix}\lambda_1=0 & (-2,0,3)\\ \lambda_2=1 & (0,1,0)\\ \lambda_3=7 & (1,0,2)\end{matrix}\ ;\quad \Lambda=\begin{bmatrix}0&0&0\\0&1&0\\0&0&7\end{bmatrix};\quad P=\begin{bmatrix}-2&0&1\\0&1&0\\3&0&2\end{bmatrix}$$

$$A\cdot P=\begin{bmatrix}3&0&2\\0&1&0\\6&0&4\end{bmatrix}\cdot\begin{array}{c}\begin{bmatrix}-2&0&1\\0&1&0\\3&0&2\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow\\ H_1&H_2&H_3\end{matrix}\end{array}=\begin{array}{c}\begin{bmatrix}0&0&7\\0&1&0\\0&0&14\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow\\ A\cdot H_1&A\cdot H_2&A\cdot H_3\\ \parallel&\parallel&\parallel\\ \lambda_1\cdot H_1&\lambda_2\cdot H_2&\lambda_3\cdot H_3\end{matrix}\end{array}$$

$$P\cdot\Lambda=\begin{array}{c}\begin{bmatrix}-2&0&1\\0&1&0\\3&0&2\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow\\ H_1&H_2&H_3\end{matrix}\end{array}\cdot\begin{bmatrix}0&0&0\\0&1&0\\0&0&7\end{bmatrix}=\begin{array}{c}\begin{bmatrix}0&0&7\\0&1&0\\0&0&14\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow\\ 0\cdot H_1&1\cdot H_2&7\cdot H_3\end{matrix}\end{array}$$

Quindi, si ha che $\boxed{A\cdot P=P\cdot\Lambda}$

---

$$B=\begin{bmatrix}2&-1&1\\0&3&-1\\2&1&3\end{bmatrix}\quad \begin{array}{ll}\lambda_1=2 & x(-1,1,1)\ \forall x\neq 0\\ \lambda_2=4 & (1,-1,1)\end{array};\ \Lambda=\begin{bmatrix}2&0&0\\0&2&0\\0&0&4\end{bmatrix};\ P=\begin{bmatrix}-1&-5&1\\1&5&-1\\1&5&1\end{bmatrix}$$

$$P_B(\lambda)=(\lambda-2)^2\cdot(4-\lambda)$$

$$B\cdot P=\begin{bmatrix}2&-1&1\\0&3&-1\\2&1&3\end{bmatrix}\cdot\begin{bmatrix}-1&-5&1\\1&5&-1\\1&5&1\end{bmatrix}=\begin{bmatrix}-2&-10&4\\2&10&-4\\2&10&4\end{bmatrix}$$

$$P\cdot\Lambda=\begin{bmatrix}-1&-5&1\\1&5&-1\\1&5&1\end{bmatrix}\cdot\begin{bmatrix}2&0&0\\0&2&0\\0&0&4\end{bmatrix}=\begin{bmatrix}-2&-10&4\\2&10&-4\\2&10&4\end{bmatrix}$$

Quindi, si ha che $\boxed{B\cdot P=P\cdot\Lambda}$

---

$$C=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}\quad \begin{array}{ll}\lambda_1=1 & (1,0,-1),(0,1,-1)\\ \lambda_2=7 & (1,2,3)\end{array}\quad \Lambda=\begin{bmatrix}1&0&0\\0&7&0\\0&0&1\end{bmatrix};\ P=\begin{bmatrix}1&1&0\\0&2&1\\-1&3&-1\end{bmatrix}$$

$$P_C(\lambda)=(\lambda-1)^2\cdot(7-\lambda)$$

$$C\cdot P=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}\cdot\begin{bmatrix}1&1&0\\0&2&1\\-1&3&-1\end{bmatrix}=\begin{bmatrix}1&7&0\\0&14&1\\-1&21&-1\end{bmatrix}$$

$$P\cdot\Lambda=\begin{bmatrix}1&1&0\\0&2&1\\-1&3&-1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&7&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&7&0\\0&14&1\\-1&21&-1\end{bmatrix}$$

Quindi, si ha che $\boxed{C\cdot P=P\cdot\Lambda}$

---

**DEFINIZIONE** Sia $A_{n\times n}$ una matrice quadrata di ordine $n$. Diremo che

$A$ è **DIAGONALIZZABILE** se esiste una matrice quadrata di ordine $n$ diagonale $\Lambda$ ed esiste una matrice quadrata di ordine $n$ invertibile $P$ tali che

$$A = \underbrace{P \cdot \Lambda \cdot P^{-1}}_{\text{forma spettrale di } A}$$

---

osservazione: se $A$ è già in partenza una matrice diagonale $A = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$, allora $A$ è anche diagonalizzabile.

$\Lambda = A$ ; $\quad P = I_n \quad P^{-1} = I_n$

$$A = I_n \cdot A \cdot I_n = P \cdot \Lambda \cdot P^{-1}$$

---

$A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 6 & 0 & 4 \end{bmatrix}$; $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$; $A \cdot P = P \cdot \Lambda$

$\det P \neq 0 \Rightarrow \exists P^{-1}$

$B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$; $\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $P = \begin{bmatrix} -1 & -5 & 1 \\ 1 & 5 & -1 \\ 1 & 5 & 1 \end{bmatrix}$; $B \cdot P = P \cdot \Lambda$

$\det P = 0 \Rightarrow \nexists P^{-1}$

$C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$; $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 3 & -1 \end{bmatrix}$; $C \cdot P = P \cdot \Lambda$

$\det P \neq 0 \Rightarrow \exists P^{-1}$

$A \cdot P = P\Lambda$ et $\exists P^{-1} \Rightarrow (A \cdot P) \cdot P^{-1} = (P \cdot \Lambda) P^{-1} \Rightarrow$

$\Rightarrow A \cdot (P \cdot P^{-1}) = P \cdot \Lambda \cdot P^{-1} \Rightarrow A \cdot I_n = P \cdot \Lambda \cdot P^{-1} \Rightarrow$

$\Rightarrow A = P \cdot \Lambda \cdot P^{-1}$

$B \cdot P = P \cdot \Lambda$ ma $\nexists P^{-1}$

$C \cdot P = P \cdot \Lambda$ et $\exists P^{-1} \Rightarrow \dots\dots \Rightarrow C = P \cdot \Lambda \cdot P^{-1}$

---

$\boxed{\text{TEOREMA}}$ (senza dimostrazione)

Una matrice quadrata $A$ di ordine $n$ è **diagonalizzabile** <u>se e solo se</u> esiste una base di $\mathbb{R}^n$ formata da autovettori della matrice $A$.

---

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}; \quad P_D(\lambda) \overset{\text{DEF.}}{=\!=} \det \underbrace{\begin{bmatrix} (1-\lambda) & 0 & 1 & 0 \\ 1 & (1-\lambda) & 0 & 1 \\ 1 & 0 & (1-\lambda) & 0 \\ 0 & 1 & 1 & (1-\lambda) \end{bmatrix}}_{B}$$

$b_{11} = 1-\lambda$

$b_{12} = 0$

$b_{13} = 1$

$b_{14} = 0$

$$P_D(\lambda) = \sum_{j=1}^{4} (-1)^{1+j} \cdot b_{1j} \cdot \det B_{1j} = (-1)^{1+1} \cdot (1-\lambda) \cdot \det \begin{bmatrix} (1-\lambda) & 0 & 1 \\ 0 & (1-\lambda) & 0 \\ 1 & 1 & (1-\lambda) \end{bmatrix} +$$

$$+ (-1)^{1+3} \cdot 1 \cdot \det \begin{bmatrix} 1 & (1-\lambda) & 1 \\ 1 & 0 & 0 \\ 0 & 1 & (1-\lambda) \end{bmatrix} =$$

$$= (1-\lambda)\cdot\left\{(1-\lambda)\cdot\left[(1-\lambda)^2-1\right]\right\} + 1\cdot\left\{-1\cdot\left[(1-\lambda)^2-1\right]\right\} =$$

$$= (1-\lambda)^2\cdot\left[1+\lambda^2-2\lambda-1\right] - \left[1+\lambda^2-2\lambda-1\right] =$$

$$= (1-\lambda)^2\cdot(\lambda^2-2\lambda) - (\lambda^2-2\lambda) = (\lambda^2-2\lambda)\cdot\left[(1-\lambda)^2-1\right] =$$

$$= (\lambda^2-2\lambda)\cdot(\lambda^2-2\lambda) = (\lambda^2-2\lambda)^2 = \left[\lambda\cdot(\lambda-2)\right]^2 = \lambda^2\cdot(\lambda-2)^2$$

$$P_D(\lambda) = \lambda^2\cdot(\lambda-2)^2 \qquad \lambda_1=0 \quad m_a(0)=2$$

$$\lambda_2=2 \quad m_a(2)=2$$

$\boxed{\lambda_1=0} \longrightarrow$ autovettori ? $\quad D-0\cdot I = D$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \bullet \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} w+y=0 \\ w+x+z=0 \\ \cancel{w+y}=0 \\ x+y+z=0 \end{cases} \qquad \begin{cases} y=-w \\ w+x+z=0 \\ x-w+z=0 \end{cases}$$

$$\begin{cases} y=-w \\ w+x+z=0 \\ 2w=0 \end{cases} \qquad \begin{cases} w=0 \\ y=0 \\ z=-x \end{cases} \qquad (w,x,y,z)=(0,x,0,-x)=$$

$$= x\cdot(0,1,0,-1)$$

autovettori relativi a $\lambda_1=0$ $\quad x\cdot(0,1,0,-1) \quad \forall x\in\mathbb{R}-\{0\}$

$$m_g(0)=1$$

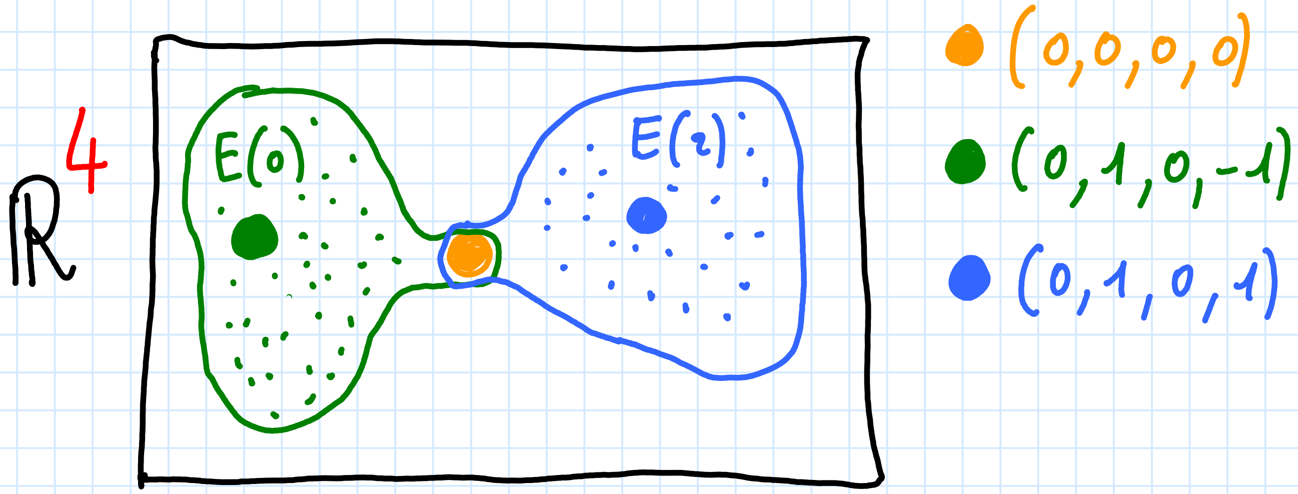
$\boxed{\lambda_2=2} \longrightarrow$ autovettori ? $\quad D-2\cdot I$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -w + y = 0 \\ w - x + z = 0 \\ w \cancel{-} y = 0 \\ x + y - z = 0 \end{cases} \qquad \begin{cases} y = w \\ w - x + z = 0 \\ x + w - z = 0 \end{cases}$$

$$\begin{cases} y = w \\ w - x + z = 0 \\ 2w = 0 \end{cases} \qquad \begin{cases} w = 0 \\ y = 0 \\ z = x \end{cases} \qquad (w,x,y,z) = (0,x,0,x) = \\ = x \cdot (0,1,0,1)$$

autovettori relativi a $\lambda_2 = 2$ $\quad x \cdot (0,1,0,1) \quad \forall x \in \mathbb{R} - \{0\}$

$$m_g(2) = 1$$




Ho <u>infiniti</u> autovettori. Ma ne riesco a prendere AL MASSIMO due linearmente INDIP.

Quindi, <u>NON</u> esiste una base di $\mathbb{R}^4$ formata da autovettori di D.

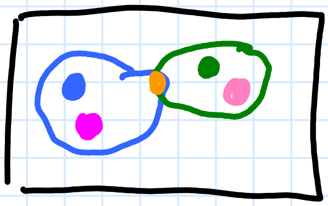
Per cui, D <u>NON</u> è diagonalizzabile.

---

$E_{4\times 4} = \begin{bmatrix} \quad & \quad \end{bmatrix}$ ; $\quad p_E(\lambda) = \lambda^2 \cdot (\lambda - 2)^2$

$\lambda_1 = 0 \qquad \lambda_2 = 2$

$\mathbb{R}^4$



ESISTE base di $\mathbb{R}^4$ formata da autovettori di $E$

Quindi, $E$ è diagonalizzabile